Northeast. Math. J.
17(1) (2001) , 5 –8

Monotone Iterative Method for Set-valued Quasilinear Elliptic Boundary Value Problems

SUN Le-lin (孙乐林) , XU Di-hong (徐迪红) and CHENG Jian (程　建)
(School of Mathematical Sciences , Wuhan University , Wuhan , 430072*)*

Communicated by Li Rong-hua (李荣华)

Key words : set-valued differential equation , elliptic boundary value problem , generalized monotone iteration

1991 MR subject classification : 65N , 35D

CLC number : O241. 82 , O175. 29

Document code : A

Article ID : 1000-1778(2001) 01-0005-04

§1. Problem and Assumptions

This paper deals with the solutions of the following differential inclusion problem:

$$\begin{cases} Au \in f(x, u), & x \in \Omega; \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where

$$Au(x) = -\sum_{i=1}^{N} D_i[a_i(x, Du(x))],$$

$\Omega \subset R^N$ is a bounded domain with piecewise Lipschitz boundary $\partial\Omega$, $Du = (D_1 u, D_2 u, \cdots, D_N u)$, $D_i u = \frac{\partial u}{\partial x_i}$, $i = 1, 2, \cdots, N$, and $f: \Omega \times R \to 2^R$ is a set-valued function.

Let $p \geq 2$ be an integer and q its conjugate exponent, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Let $W^{1,p}(\Omega)$ denote the usual Sobolev space and $(W^{1,p}(\Omega))^*$ its dual space, $\langle \cdot, \cdot \rangle$ the duality pairing between the elements of $(W^{1,p}(\Omega))^*$ and $W^{1,p}(\Omega)$, $\| \cdot \|_{1,p}$ and $\| \cdot \|_p$ the norms in $W^{1,p}(\Omega)$ and in $L^p(\Omega)$ respectively. A partial ordering is defined in $L^p(\Omega)$ and in $W^{1,p}(\Omega)$ by $u \leq v$ if and only if $v - u$ belongs to the set $L_+^p(\Omega)$ of all nonnegative elements of $L^p(\Omega)$.

Received date : June 28 , 2000.

Foundation item : The NSF (19771062) of China.

We impose the standard conditions of Leray-Lions type (cf. e. g. , [1] and [2]) on the functions $a_i: \Omega \times R^N \to R$, $i = 1, 2, \cdots, N$, and the following hypotheses on the function f:

(H1) The function $\xi \mapsto f(x, \xi)$ is upper semicontinuous for a. a. $x \in \Omega$, and $f(x, \xi)$ is compact-valued and convex-valued for a. a. $x \in \Omega$ and for all $\xi \in W^{1,p}(\Omega)$;

(H2) $f(x, \xi)$ is an order monotone function with respect to $\xi \in W^{1,p}(\Omega)$, i. e. , $\forall (\xi, r) \in$ graph(f), $\forall (\eta, s) \in$ graph(f), $r \leq s$ if $\xi \leq \eta$;

(H3) There exists a single-valued function $g(x) \in L^q_+(\Omega)$ such that $|f(x, \xi)| \leq g(x)$ for a. a. $x \in \Omega$.

We define the semilinear form

$$l(u, v) = \sum_{i=1}^{N} \int_\Omega a_i(x, Du) D_i v \mathrm{d}x$$

on $W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ and denote that

$$\Phi_u = \{ w \in L^q(\Omega) : w(x) \in f(x, u(x)), \ u \in W_0^{1,p}(\Omega) \}. \tag{2}$$

A function $u \in W_0^{1,p}(\Omega)$ is said to be a weak solution of problem (1) if there exists a function $w \in \Phi_u$ such that

$$l(u, v) = \int_\Omega wv \mathrm{d}x, \quad \forall v \in W_0^{1,p}(\Omega). \tag{3}$$

A function $\bar{u} \in W_0^{1,p}(\Omega)$ is an upper solution of (1) if

$$l(\bar{u}, v) \geq \int_\Omega wv \mathrm{d}x, \quad \forall w \in \Phi_{\bar{u}}, \ \forall v \in W_0^{1,p}(\Omega) \cap L^p_+(\Omega).$$

A lower solution is defined similarly by reversing the above inequality.

For any $u \in W^{1,p}(\Omega)$, we define an operator $L: W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^* \cong W^{-1,q}(\Omega)$ by

$$\langle Lu, v \rangle = l(u, v), \quad \forall v \in W_0^{1,p}(\Omega) \tag{4}$$

and a set-valued operator $K: W_0^{1,p}(\Omega) \to 2^{W^{-1,q}(\Omega)}$ by

$$Ku = \left\{ y \in W^{-1,q}(\Omega) : \exists\, w \in \Phi_u, \text{ s.t. } \langle y, v \rangle = \int_\Omega wv \mathrm{d}x, \ \forall v \in W_0^{1,p}(\Omega) \right\}. \tag{5}$$

Therefore, the variational problem of (1) can be described by operator form as follows:

Find $u \in W_0^{1,p}(\Omega)$ such that

$$Lu \in Ku. \tag{6}$$

§2. Generalized Monotone Iteration and Existence Theorem

We approximate a solution of (6) with the following iteration:

Step 1 Find $\bar{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$l(\bar{u}_0, v) = \int_\Omega gv \mathrm{d}x, \quad \forall v \in W_0^{1,p}(\Omega), \tag{7}$$

where g satisfies (H3). The equation (7) is solvable (cf. [2], P. 341). And by means of (H3)

$\bar{u}_0$ is an upper solution of (1).

Step 2 Choose a function $\bar{w}_k \in \Phi_{\bar{u}_k}$ and find $\bar{u}_{k+1} \in W_0^{1,p}(\Omega)$ such that

$$l(\bar{u}_{k+1}, v) = \int_\Omega \bar{w}_k v \,\mathrm{d}x, \quad \forall v \in W_0^{1,p}(\Omega), \ k = 0, 1, 2, \cdots \tag{8}$$

Using the conditions given in the above section and some related results in [1], [3] and [4], we can prove the following lemmas and theorem:

Lemma 1 *Let* $u, v \in W^{1,p}(\Omega)$ *and denote* $u^+ = \max\{u, 0\}$. *Then*

$$\langle Lu - Lv, (u - v)^+ \rangle \geq 0.$$

Theorem 1 *The two sequence* $\{\bar{u}_k\}$ *and* $\{\bar{w}_k\}$ *generated by the iterative procedure* (7)—(8) *are both nonincreasing.*

Lemma 2 $\{\bar{u}_k\}$ *and* $\{\bar{w}_k\}$ *are uniformly bounded in* $W_0^{1,p}(\Omega)$ *and in* $L^q(\Omega)$ *respectively.*

Lemma 3([4], Lemma 4.22) *Let the sequence* $\{u_k\}$ *converge to* u *weakly in* $W^{1,p}(\Omega)$ *and* $l(u_k, u_k - u) - l(u, u_k - u) \to 0$ *as* $k \to \infty$. *Then* $\{u_k\}$ *converges to* u *strongly in* $W^{1,p}(\Omega)$.

Now, let $\underline{u}_0$ be a solution of the following equation:

$$l(u, v) = -\int_\Omega g v \,\mathrm{d}x, \quad \forall v \in W_0^{1,p}(\Omega).$$

It is clear that $\underline{u}_0$ is a lower solution of (1). Denote that $C_u = \{\bar{u}_k\}$ and $C_w = \{\bar{w}_k\}$. They are nonincreasing well ordered chains in $[\underline{u}_0, \bar{u}_0]_{W^{1,p}(\Omega)}$ and in $[\underline{w}_0, \bar{w}_0]_{L^q(\Omega)}$ respectively, where

$$[\underline{u}_0, \bar{u}_0]_{W^{1,p}(\Omega)} = \{u \in W^{1,p}(\Omega) : \underline{u}_0 \leq u \leq \bar{u}_0\}$$

and

$$[\underline{w}_0, \bar{w}_0]_{L^q(\Omega)} = \{w \in L^q(\Omega) : \underline{w}_0 \leq w \leq \bar{w}_0\}.$$

The following are main conclusions of this paper.

Lemma 4 *The sequence* $\{\bar{u}_k\}$ *converges to* $u^* = \inf C_u$ *weakly in* $W_0^{1,p}(\Omega)$ *and strongly in* $L^p(\Omega)$, *and the sequence* $\{\bar{w}_k\}$ *converges to* $w^* = \inf C_w$ *strongly in* $L^q(\Omega)$.

Theorem 2 *The function* $u^* = \inf C_u$ *is a solution of problem* (6).

Proof By using the results of above lemmas we can deduce that

$$\langle Lu^*, v \rangle = \int_\Omega w^* v \,\mathrm{d}x, \quad \forall v \in W_0^{1,p}(\Omega). \tag{9}$$

What remaines is to prove $w^* \in \Phi_{u^*}$. For this purpose, we define a support function $\sigma(f(x, u), v)$ (cf. [5], Ch. 3) associated with $f(x, u)$ as follows:

$$\sigma(f(x, u), v) = \sup_{w \in \Phi_u} \int_\Omega w v \,\mathrm{d}x, \quad \forall v \in W_0^{1,p}(\Omega).$$

So, we have

$$\int_\Omega \bar{w}_k v \,\mathrm{d}x \leq \sigma(f(x, \bar{u}_k), v), \quad \forall \bar{w}_k \in \Phi_{\bar{u}_k}, \ \forall v \in W_0^{1,p}(\Omega). \tag{10}$$

From the hypothesis (H1), $f(x, u)$ is upper semicontinuous at u^*. So we might as well deem that, for any $\varepsilon > 0$, there exists an associated $\delta > 0$ such that

$$\| u^* - u \|_p < \delta \Rightarrow f(x, u) \subseteq \rho(f(x, u^*), \varepsilon), \quad \text{a.a. } x \in \Omega,$$

where

$$\rho(f(x,u^*),\varepsilon) = \{v \in L^q(\Omega) : \inf_{w \in \Phi_{u^*}} \| v - w \|_q < \varepsilon\}.$$

We then know, by means of Lemma 4, that

$$f(x,\overline{u_k}) \subseteq \rho(f(x,u^*),\varepsilon) \tag{11}$$

if k is large enough.

Taking the support function for (11) and noting (10), we have the following inequality:

$$\int_\Omega w_k v\,\mathrm{d}x \leq \sigma(f(x,\overline{u_k}),v) \leq \sigma(f(x,u^*),v) + \varepsilon, \quad \forall v \in W_0^{1,p}(\Omega).$$

By taking the limit when $k \to \infty$ and $\varepsilon \to 0$, we obtain

$$\int_\Omega w^* v\,\mathrm{d}x \leq \sigma(f(x,u^*),v), \quad \forall v \in W_0^{1,p}(\Omega).$$

Noticing the compactness and convexity of f, it can be concluded that

$$w^* \in \overline{\mathrm{cov}} f(x,u^*) = f(x,u^*).$$

Moreover, by Lemma 4, $w^* \in \Phi_{u^*}$. This result coupled with (9) shows us that u^* satisfies (3) (i.e., (6)). The proof of Theorem 2 is completed.

References

[1] Heikkilä, S. and Lakshmikantham, V., Monotone Iterative Techniques for Nonlinear Differential Equations, Marcel Dekker, Inc., New York, 1994.

[2] You, Z., Gong, H. and Xu, Z., Nonlinear Analysis (in Chinese), Xi'an Jiaotong Univ. Press, Xi'an, 1991.

[3] Gilbarg, D. and Tradinger, N. S., Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, Heidelberg, New York and Tokyo, 1977.

[4] Toianiello, G. M., Elliptic Differential Equations and Obstacle Problems, Plenum Press, New York and London, 1987.

[5] Aubin, J. P. and Ekeland, I., Applied Nonlinear Analysis, John Wiley & Sons. Inc., New York, 1984.